Decoherence

The Schrödinger equation describes the evolution of quantum systems *in isolation*. These *closed* systems have a well-defined Hamiltonian, which gives complete information about how these systems evolve. The resulting evolution, as we have seen, is unitary.

Note that these Hamiltonians may "come from outside" the system; for instance, we can turn external fields on and off, shine lasers, etc. What makes a quantum system closed is that it doesn't act back on the external world. The external fields, lasers, etc., can all be treated classically.

The unfortunate reality is that this idealization is a fiction. All realistic quantum systems interact with the outside world at least weakly; and the existence of interactions which allow us to manipulate the systems also allows the systems to interact with the external environment. This environmental interaction is called *decoherence*.

Two things happen in decoherence. First, random influences from the outside can perturb our system's evolution. This is as if some random Hamiltonian was turned on, in addition to the usual Hamiltonian.

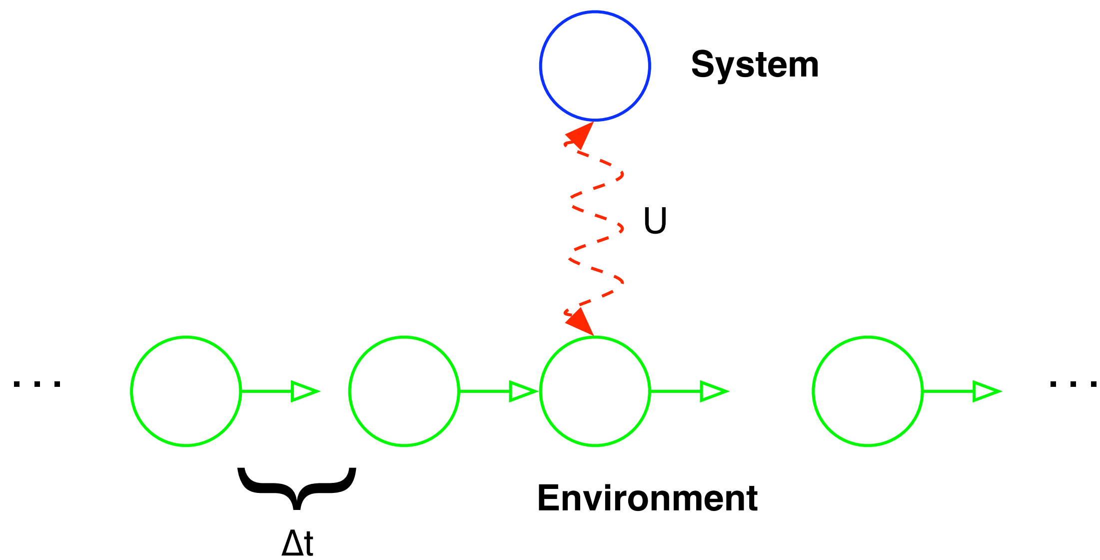
Second, the interaction between the system and environment can leave information about the system stored in the environment; this is called a *generalized record*. The effect on the system is as if unwanted measurements have been performed (without, in general, our knowing the measurement results).

In fact, these two processes generally both occur; and the practical effects of them often look similar. Indeed, in quantum mechanics there is no sharp distinction between them, as we shall see.

Let us look at a very simple, schematic picture of how decoherence works. This will give us understanding of the type of effects we have to deal with.

A Simple Model

For simplicity, we will let our quantum system be a single q-bit. The environment consists of a stream of q-bits, which approach the system one at a time, interact briefly, and then fly off, never to be seen again.



We assume that these environment bits do not interact with each other. The average time between environment q-bits is Δt. The interaction results in a joint unitary $\hat{U}$ being done on the two q-bits.

This unitary transformation $\hat{U}$ could be practically anything, depending on the nature of the physical system and the environment. For a first look, however, let's consider a very familiar two-bit unitary: the CNOT.

Let's first have the CNOT go from the environment bit onto the system bit. Suppose the system is initially in a pure state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, and that the environment bits all start in the state $|0\rangle$. Then the system state will be completely unchanged after interacting with the environment:

$$(\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle \rightarrow (\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle.$$

Suppose, however, that a small proportion p of the environment bits are in state $|1\rangle$. If the environment bit is in state $|1\rangle$,

$$(\alpha|0\rangle + \beta|1\rangle) \otimes |1\rangle \rightarrow (\alpha|1\rangle + \beta|0\rangle) \otimes |1\rangle.$$

This is a *bit-flip error*.

If the environment bits have probability p of being in state $|1\rangle$ and $1-p$ of being in $|0\rangle$, we usually have no way of knowing which state any particular bit is in. We therefore describe the environment bits by a *mixed state*

$$\rho_E = (1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|.$$

What happens when a bit in this state does a CNOT on the system bit in state $|\psi\rangle\langle\psi|$?

$$\begin{aligned}|\psi\rangle\langle\psi| \otimes \rho_E \rightarrow &(1-p)|\psi\rangle\langle\psi| \otimes |0\rangle\langle 0| \\ &+ p\hat{X}|\psi\rangle\langle\psi|\hat{X} \otimes |1\rangle\langle 1|.\end{aligned}$$

This is a separable state, but not a product state. After the environment bit flies off, we describe the system by its reduced density matrix by tracing out the environment bit. The net effect on the system bit is the completely positive transformation:

$$|\psi\rangle\langle\psi| \rightarrow (1-p)|\psi\rangle\langle\psi| + p\hat{X}|\psi\rangle\langle\psi|\hat{X}.$$

It is as if a gate $\hat{X}$ is applied to the system with probability p, but we have no way of knowing whether or not this has actually happened. This is an example of the first kind of decoherence mentioned above: random influences acting on the state.

We can write this completely positive transformation as $\mathcal{O} = (1-p)\mathcal{I} + p\mathcal{X}$, where $\mathcal{O}$, $\mathcal{I}$ and $\mathcal{X}$ represent *superoperators*:

$$\mathcal{I}\rho = \rho, \quad \mathcal{X}\rho = \hat{X}\rho\hat{X},$$

$$\mathcal{O}\rho = (1-p)\rho + p\hat{X}\rho\hat{X}.$$

After interacting with n environment bits, the reduced density matrix of the system is $\rho_n = \mathcal{O}^n\rho$. In the limit of large n,

$$\rho_n \rightarrow (1/2)(|\psi\rangle\langle\psi| + \hat{X}|\psi\rangle\langle\psi|\hat{X}).$$

How big a change this is depends on how close $|\psi\rangle$ was to an $\hat{X}$ eigenstate.

For the second kind of decoherence, let us turn the CNOT around and have it act from the system bit onto the environment bit. Suppose for the moment that all the environment bits start in state $|0\rangle$. What happens then?

$$(\alpha|0\rangle + \beta|1\rangle)|0\rangle \to \alpha|0\rangle|0\rangle + \beta|1\rangle|1\rangle.$$

If the system interacts with n environment q-bits, the joint state of all $n+1$ bits is

$$(\alpha|0\rangle + \beta|1\rangle)|0\rangle^{\otimes n} \to \alpha|0\rangle|0\rangle^{\otimes n} + \beta|1\rangle|1\rangle^{\otimes n}.$$

In either case, the reduced density matrix of the system becomes

$$|\psi\rangle\langle\psi| \to |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|.$$

It is as if the system was measured in the $|0\rangle, |1\rangle$ basis without our knowing the outcome of the measurement. However, the outcome does exist—it is stored in the environment bits. We could find it out (in principle) by measuring the environment, an example of an indirect measurement.

In this case, the superoperator is

$$\mathcal{O}\rho = |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|.$$

In fact, the CNOT is a rather strong interaction. As we have seen, interacting with a single environment bit is enough to *decohere* the system completely; the additional environment bits add nothing to the effect.

In fact, because experimenters go to great lengths to isolate the systems they work with as well as possible, the interactions between system and environment are usually *weak*. By weak, I mean that they alter the system state little; the unitary $\hat{U}$ is close to the identity.

Consider the family of two-bit gates which produce a controlled X rotation:

$$\hat{U}(\theta) = |0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{R}_x(\theta).$$

This family includes a gate similar to the CNOT ($\theta = \pi$), but we are interested in members with $\theta \ll 1$. In this case, after the interaction the joint state is

$$\alpha|0\rangle|0\rangle + \beta|1\rangle(\cos(\theta/2)|0\rangle - i\sin(\theta/2)|1\rangle).$$

The reduced density matrix of the system is

$$|\psi\rangle\langle\psi| \to |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|$$

$$+\cos(\theta/2)\left(\alpha\beta^*|0\rangle\langle 1| + \alpha^*\beta|1\rangle\langle 0|\right)$$

$$\approx |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|$$

$$+(1-\theta^2/8)\left(\alpha\beta^*|0\rangle\langle 1| + \alpha^*\beta|1\rangle\langle 0|\right).$$

After interacting with n environment bits, the diagonal terms of the reduced density matrix are unchanged, but the off-diagonal terms go to

$$(1-\theta^2/8)^n\,(\alpha\beta^*|0\rangle\langle 1| + \alpha^*\beta|1\rangle\langle 0|)$$
$$\approx \exp(-n\theta^2/8)\,(\alpha\beta^*|0\rangle\langle 1| + \alpha^*\beta|1\rangle\langle 0|)\,.$$

The off-diagonal terms decay away exponentially with time! So the end result of this weak decoherence process (at long times) is the same as the strong decoherence using the full CNOT: the environment bits have effectively measured the system.

In this case as well, the outcome of the measurement is stored in the environment. This time, though, measuring a single environment bit wouldn't be enough in general to learn the outcome; we would have to measure all the bits that had interacted with the system, and if *none* of them is in state $|1\rangle$ then (with high probabililty) the system is in state $|0\rangle$.

Direction of influence

We have separated decoherence into two kinds—random noise entering from the environment, and "measurement" of the system producing generalized records in the environment. As we have seen already, however, the idea that there is a fixed direction in which information flows is not compatible with the laws of quantum mechanics. A CNOT can be "flipped" into the other direction by changing from the Z to the X basis, for instance; and this two-way character of quantum interactions is true for all other quantum gates.

Suppose that instead of $|0\rangle, |1\rangle$, the environment bits are in $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ with probabilities $(1-p)$ and p. Then a CNOT onto the environment results in

$$(\alpha|0\rangle + \beta|1\rangle)|\pm\rangle \rightarrow (\alpha|0\rangle \pm \beta|1\rangle)|\pm\rangle.$$

If the environment bit is in the $|-\rangle$ state, this results in a *phase flip* error on the system.

System evolution

After each environment bit interacts with the system, it is lost and never returns to interact again. The best we can hope to do is describe the reduced density matrix of the system, obtained by tracing out the environment.

After interacting with a single environment bit, the reduced density matrix of the system becomes

$$\rho \to \sum_k \hat{A}_k \rho \hat{A}_k^\dagger$$

for some set of operators $\{\hat{A}_k\}$ such that

$$\sum_k \hat{A}_k^\dagger \hat{A}_k = \hat{I}.$$

We have seen this before: it is a *trace-preserving completely positive map.* Such a decomposition into operators is called a *Kraus representation.*

In general, the choice of operators $\{\hat{A}_k\}$ will not be unique. Many sets of operators can correspond to the same completely positive map. For instance, consider our bit-flipping environment:

$$\rho \rightarrow (1-p)\rho + p\hat{X}\rho\hat{X}.$$

One choice of operators would be

$$\hat{A}_1 = \sqrt{1-p}\hat{I}, \quad \hat{A}_2 = \sqrt{p}\hat{X}.$$

However, it is easy to see that the following operators also work:

$$\hat{A}'_{1,2} = \sqrt{(1-p)/2}\hat{I} \pm \sqrt{p/2}\hat{X}.$$

In fact, for a given completely positive map, there are infinitely many Kraus representations.

The Markov Assumption

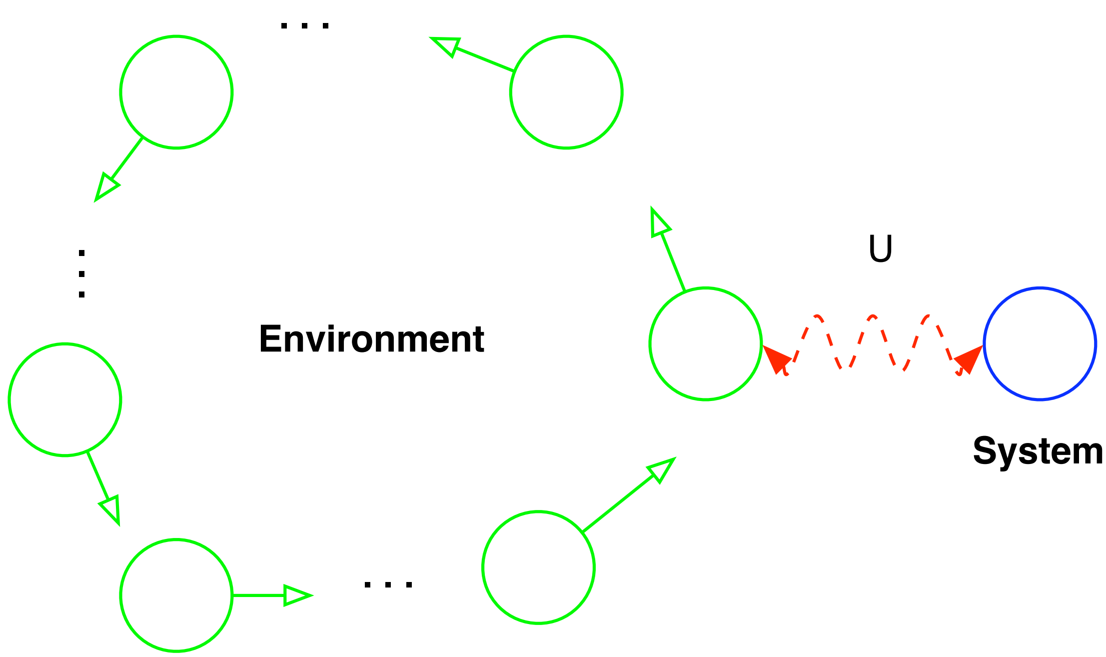
Note that there is an important feature to our model of the environment. After each environment bit interacts with the system, it never acts on the system again.

This means that as far as the system is concerned, it is fine to just trace out the departed bits and forget about them. This is what makes the evolution of the system take the simple Kraus form.

$$\rho \to \rho' = \sum_k \hat{A}_k \rho \hat{A}_k^\dagger .$$

We call such an evolution *Markovian* or *time-local*, meaning that the new state ρ' depends only on the state ρ at the present time.

This need not, however, be the case. Suppose that the system interacted with the environment bits repeatedly:



In this case, the state of the environment bits has been influenced by the system state at earlier times. This would make the evolution of the system state in the present dependent on its value in the past. It is no longer correct to simply trace out the environment bits.

One could get around this problem by describing the system and environment together. Unfortunately, this is often even more difficult! The environment is in general very large; the system and environment together have an enormous Hilbert space. Describing a state in this space is very difficult, especially since we will often not know the exact initial state of the environment. Moreover, we are not usually interested in the state of the environment.

Fortunately, many environments are effectively Markovian. And even when this is not exactly true, it is often a reasonable approximation. For the purposes of this course, we will always assume that decoherence is a Markovian process.

The master equation

This description of evolution by completely positive maps uses a discrete time variable. It is as if we are taking snapshots of the system after each environmental interaction.

In the case of unitary evolution, the unitary transformations arose from a continuous equation: the Schrödinger equation

$$i\hbar\frac{d|\psi\rangle}{dt} = \hat{H}|\psi\rangle.$$

For density matrices, this becomes the von Neumann equation:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[\hat{H},\rho].$$

Both of these produce unitary evolution. Are there continuous equations which yield completely positive evolutions?

Indeed there are such equations; they are called *master equations*. The most general Markovian master equation that produces completely positive evolution is

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[\hat{H}, \rho]$$

$$+ \sum_{k=1}^{n} 2\hat{L}_k \rho \hat{L}_k^\dagger - \hat{L}_k^\dagger \hat{L}_k \rho - \rho \hat{L}_k^\dagger \hat{L}_k,$$

where the $\{\hat{L}_k\}$ are a set of n operators which represent the effects of the environment. An equation of this type is said to be in *Lindblad form*, and the operators $\{\hat{L}_k\}$ are often called *Lindblad operators*.

The right-hand side of this equation is often written as a superoperator:

$$\frac{d\rho}{dt} = \mathcal{L}\rho.$$

This superoperator $\mathcal{L}$ is sometimes called the *Liouvillian*.

We can see that this is completely positive by re-writing it

$$\rho(t) \rightarrow \rho(t + \Delta t) = \sum_{k=1}^{n+1} \hat{A}_k \rho \hat{A}_k^\dagger,$$

where the operators $\hat{A}_k$ are

$$\hat{A}_k = \sqrt{2\Delta t}\hat{L}_k. \quad k = 1, \ldots, n,$$

$$\hat{A}_{n+1} = \hat{I} - \Delta t \left(i\hat{H}/\hbar + \sum_{k=1}^{n} \hat{L}_k^\dagger \hat{L}_k \right).$$

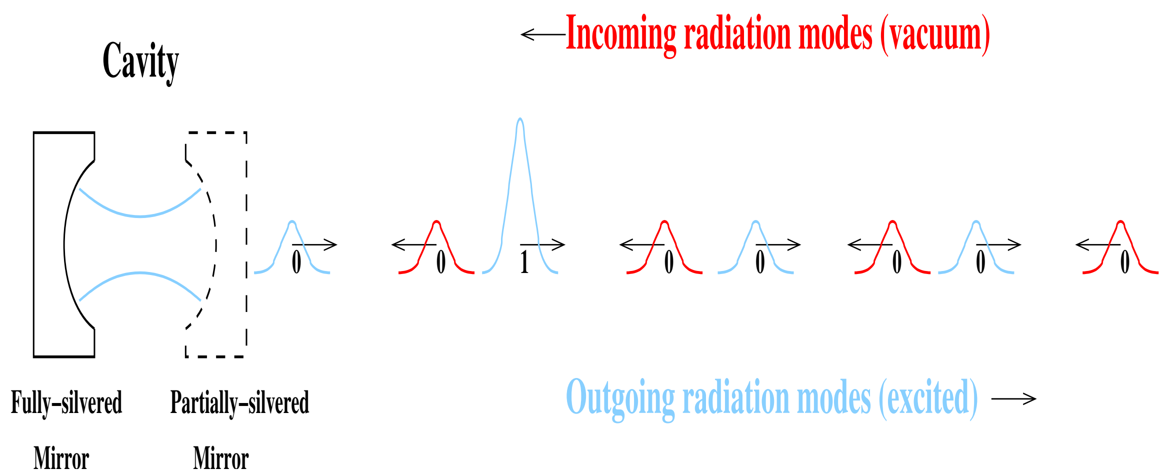
Up to $O(\Delta t^2)$ this satisfies

$$\frac{\rho(t + \Delta t) - \rho(t)}{\Delta t} = \mathcal{L}\rho(t),$$

$$\sum_{k=1}^{n+1} \hat{A}_k^\dagger \hat{A} = \hat{I}.$$

Physical environments

While this q-bit model of decoherence is highly idealized and simplified, it does represent processes which can occur in a real physical system. For instance, consider the following optical system:



Of course, most environments don't look much like a stream of bits. Every physical system has its own appropriate environment. Here are a few examples:

Ions in traps undergo *spontaneous decay* due to interaction with the external electromagnetic field; they also undergo *heating* due to nonuniformities in the magnetic field.

Solid state q-bits interact with *phonons*, which are vibrational modes of the solids.

Electron and nuclear spins interact with other nearby spins by spin-spin dipole interactions.

Photons in cavities can leak out into the outside electromagnetic field; they can also be absorbed by imperfect mirrors, or scattered by dust.

To understand how well a quantum system will work for QIP, we need to understand its particular decoherence processes. The most important quantity is the *rate* of decoherence: how rapidly a pure state become mixed due to interaction with the environment.

While decoherence is different for different systems, the bottom line is the same: decoherence destroys QIP by distorting the desired unitary evolution. In other words, decoherence produces *errors*, and we need to learn how to cope with them.

Next time: error models in QIP.