

Measurement and Interference

Let us recall what we learned about projective measurements. Any ideal physical measurement on a system with a D -dimensional Hilbert space corresponds to an *observable*, which is a $D \times D$ Hermitian operator $\hat{O} = \hat{O}^\dagger$. Such operators are diagonalizable:

$$\hat{O} = \sum_j \lambda_j \hat{\mathcal{P}}_j,$$

with $M \leq D$ real distinct eigenvalues λ_j and M corresponding orthogonal projectors $\hat{\mathcal{P}}_j = \hat{\mathcal{P}}_j^\dagger$, $\hat{\mathcal{P}}_j \hat{\mathcal{P}}_k = \hat{\mathcal{P}}_j \delta_{jk}$.

When a measurement is carried out, the *outcome* of the measurement (i.e., the measured value) is one of the eigenvalues λ_j , and the system is left in a corresponding eigenstate of \hat{O} . This implies that if the measurement is immediately repeated, the same outcome will occur.

The probability of a particular outcome is determined by *Born's rule*. Let's see how this works for an observable that is *nondegenerate*, i.e., has $M = D$ distinct real eigenvalues λ_j . In this case, each of the projectors $\hat{\mathcal{P}}_j = |\phi_j\rangle\langle\phi_j|$ is one-dimensional, so the eigenstates $|\phi_j\rangle$ are uniquely determined (up to a phase) and form an orthonormal basis.

We can write any state $|\psi\rangle$ in this basis:

$$|\psi\rangle = \sum_{j=1}^D \alpha_j |\phi_j\rangle,$$

in which case the probability of the outcome λ_j is $p_j = |\alpha_j|^2$ and the system is left in the eigenstate $|\phi_j\rangle$ after the measurement.

Such an nondegenerate observable, with D distinct eigenvalues and one-dimensional spectral projectors, is called *complete*.

Suppose that the observable \hat{A} is degenerate, i.e., at least one of the projectors $\hat{\mathcal{P}}_j$ corresponding to an eigenvalue λ_j is more than one-dimensional, $\text{Tr}\{\hat{\mathcal{P}}_j\} \equiv d_j > 1$. In this case, the basis of eigenvectors is not unique. What are the outcome probabilities? What state is the system left in?

Let us choose an eigenbasis $|\phi_{jk}\rangle$, where states with the label j correspond to λ_j , and

$$\hat{\mathcal{P}}_j = \sum_k |\phi_{jk}\rangle\langle\phi_{jk}|.$$

We can write the state in terms of this basis,

$$|\psi\rangle = \sum_{j=1}^M \sum_{k=1}^{d_j} \alpha_{jk} |\phi_{jk}\rangle.$$

The probability of outcome j is then $p_j = \sum_k |\alpha_{jk}|^2$, and the system is left in the state

$$|\psi_j\rangle = \sum_k (\alpha_{jk} / \sqrt{p_j}) |\phi_{jk}\rangle.$$

The important thing to note is that this outcome does *not* depend on the particular choice of orthonormal eigenbasis $|\phi_{jk}\rangle$. Choosing any eigenbasis will give the same probabilities and the same final states. We can see this by rewriting the expressions in terms of the projector $\hat{\mathcal{P}}_j$, which *is* uniquely determined:

$$p_j = \langle \psi | \hat{\mathcal{P}}_j | \psi \rangle,$$

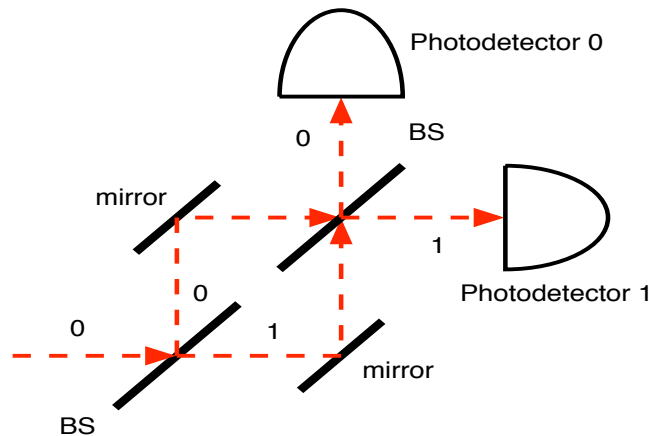
and

$$|\psi_j\rangle = \hat{\mathcal{P}}_j |\psi\rangle / \sqrt{p_j}.$$

Each eigenvalue λ_j corresponds to a *subspace* of Hilbert space; the probability of the outcome depends on the component of $|\psi\rangle$ in that subspace, and the system's state is orthogonally projected into the subspace.

Note that these expressions include the non-degenerate case, where $\hat{\mathcal{P}}_j = |\phi_j\rangle\langle\phi_j|$.

Interference occurs when amplitudes combine to augment or suppress particular outcomes:



Suppose the system is originally in the state $|0\rangle$. How does the state change as it passes through the two beam splitters?

$$\begin{aligned}
 |0\rangle &\rightarrow \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \\
 &\rightarrow \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle + \frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle = |0\rangle.
 \end{aligned}$$

Interference between the components in the two arms of the interferometer cancels the $|1\rangle$ terms while enhancing the $|0\rangle$ terms. So there will always be a click in detector 0 and never in detector 1.

What would happen if we did a projective measurement in between the two beam splitters? That is, if we measured *which path* the photon took? This would give outcomes of 0 or 1 with equal probability.

$$|0\rangle \rightarrow \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle,$$

measurement $\rightarrow |0\rangle$ or $|1\rangle$,

$$|0\rangle \rightarrow \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle,$$

$$|1\rangle \rightarrow \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle.$$

In both cases, there will be an equal probability of a click in detectors 0 or 1. So *the measurement destroys interference*.

One will often see a superposition state like $(|0\rangle + |1\rangle)/\sqrt{2}$ interpreted as meaning that the system is “either in state 0 or state 1.” But this interpretation cannot be correct. If it were, then measuring the system, to determine which state it really *was* in, should not change the outcome of *later* measurements. But it does! Measuring which arm of the interferometer the photon passed through allows detector 1 to click, when it never would without the measurement.

This effect of measurement on interference is ubiquitous in quantum mechanics. Since most quantum information protocols rely on interference effects to work, including extra measurements can destroy their effectiveness. This also includes interactions with the environment, which can effectively “measure” the system. This effect is called *decoherence*, and it is the enemy of quantum information!

What about the effect of measurement on *entanglement*? Consider the maximally-entangled state:

$$|\Psi_+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle).$$

If we measure the first system in the Z basis, we get results 0 and 1 with equal probability; after the measurement, the system is left in the product state $|00\rangle$ or $|11\rangle$. The same thing happens if we measure the second system in the Z basis.

What if we measure the first system in the X basis? We can re-write this state

$$|\Psi_+\rangle = \frac{1}{2\sqrt{2}} \left((|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) + (|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle) \right).$$

Once again, whatever the outcome, the final state will be a product.

In fact, after any *complete local measurement* (i.e., a complete measurement of one of the subsystems) the system will be left unentangled. So we see that, as with interference, measurement can destroy entanglement.

However, if we allow *joint* measurements of both subsystems, measurement can also *create* entanglement. The most obvious example of this is a *Bell state measurement*. Even if the initial state is a product $|\psi\rangle \otimes |\phi\rangle$, after the measurement the system will be in one of the Bell states:

$$|\Phi_+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$$

$$|\Phi_-\rangle = (|00\rangle - |11\rangle)/\sqrt{2}$$

$$|\Psi_+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$$

$$|\Psi_-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}.$$

In fact, *entanglement is not an observable*. There is no measurement which will reliably tell you if the pre-measurement state was entangled or not.

We can see both these effects at once in *quantum teleportation*. The initial state is $|\psi\rangle \otimes |\Psi_-\rangle$, where Alice has the first two q-bits and Bob has the third. There is no entanglement between Alice's first q-bit and the other two; but there *is* entanglement between her second q-bit and Bob's q-bit (or between Alice's joint system and Bob's q-bit).

Then Alice makes a Bell state measurement on her two q-bits together. After the measurement, there is entanglement between Alice's two q-bits; they are in one of the four maximally entangled states $|\Psi_{\pm}\rangle, |\Phi_{\pm}\rangle$. But there is no longer any entanglement between either of Alice's q-bits and Bob's.

The Bell state measurement in this case was a complete local measurement with respect to the division between Alice and Bob's systems, but an entangling joint measurement with respect to Alice's two subsystems.

Compatible measurements

As stated before, if two observables \hat{A} and \hat{B} commute,

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} = 0,$$

then \hat{A} and \hat{B} are compatible, and can be measured simultaneously. Let us see why.

By the spectral theorem, both \hat{A} and \hat{B} can be decomposed in terms of projectors:

$$\hat{A} = \sum_{j=1}^{M_A} a_j \hat{\mathcal{P}}_j, \quad \hat{B} = \sum_{k=1}^{M_B} b_k \hat{\mathcal{P}}'_k.$$

In order for $[\hat{A}, \hat{B}] = 0$ to be true, $[\hat{\mathcal{P}}_j, \hat{\mathcal{P}}'_k] = 0$ for all j, k . If two projectors $\hat{\mathcal{P}}_j$ and $\hat{\mathcal{P}}'_k$ commute, then their product is *also* a projector:

$$(\hat{\mathcal{P}}_j \hat{\mathcal{P}}'_k)^\dagger = \hat{\mathcal{P}}_j \hat{\mathcal{P}}'_k,$$

$$(\hat{\mathcal{P}}_j \hat{\mathcal{P}}'_k)^2 = (\hat{\mathcal{P}}_j)^2 (\hat{\mathcal{P}}'_k)^2 = \hat{\mathcal{P}}_j \hat{\mathcal{P}}'_k.$$

This means that it is possible to write the spectral decompositions of \hat{A} and \hat{B} in terms of more *refined* projectors:

$$\hat{A} = \sum_{j=1}^{M_A} \sum_{k=1}^{M_B} a_j \hat{\mathcal{P}}_j \hat{\mathcal{P}}'_k,$$

$$\hat{B} = \sum_{j=1}^{M_A} \sum_{k=1}^{M_B} b_k \hat{\mathcal{P}}_j \hat{\mathcal{P}}'_k.$$

(Of course, some or many of the $\hat{\mathcal{P}}_j \hat{\mathcal{P}}'_k$ may actually be zero.) We have decomposed the two operators in terms of the *same* set of projectors. We can obviously find an orthonormal basis of eigenvectors for both of these operators. Here's a simple example:

$$\hat{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

If two observables \hat{A} and \hat{B} are not compatible (i.e., do not commute), they are *complementary*; they cannot be measured simultaneously. This means that while a state $|\psi\rangle$ can be written in terms of the eigenbasis of \hat{A} or of \hat{B} , it cannot be written in terms of a simultaneous eigenbasis of both.

Our prime examples of this are the three Pauli operators \hat{X} , \hat{Y} and \hat{Z} . Each of these is an observable; they have eigenbases $\{(|0\rangle \pm |1\rangle)/\sqrt{2}\}$, $\{(|0\rangle \pm i|1\rangle)/\sqrt{2}\}$, $\{|0\rangle, |1\rangle\}$. If a spin is in one of these eigenstates, it has a definite value of the corresponding observable; but the other two observables are *completely undetermined*; any measurement of them has a 50/50 chance of being either possible result.

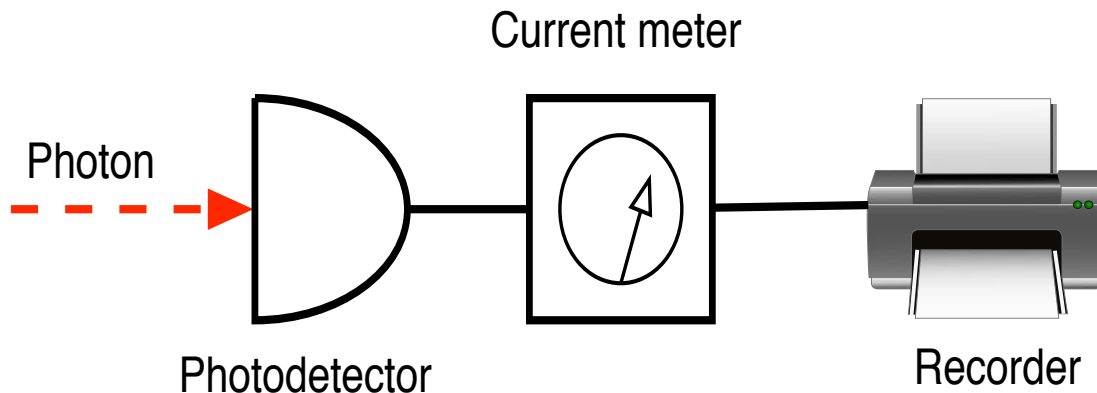
Complementarity is what makes BB84 work. Alice chooses definite values for one of two complementary variables to transmit her bits.

Experimental measurements

The mathematical formulation of projective measurement is extremely elegant. In this idealized description, measurements disturb the system as little as possible; they are repeatable, and always yield definite outcomes. In one commonly used phrase, they are *non-demolition measurements*.

In the real world, things are not so nice.

A good example of this is the *photodetector*. While there are various kinds of photodetectors, they for the most part work in roughly the same way. The photon is absorbed by the detector, the energy liberating some electric charges which produce a measurable current (called the *photocurrent*), which is detected by a current meter.

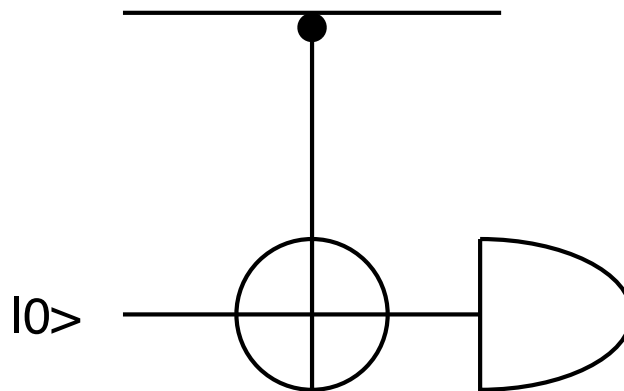


Unfortunately, this is very far from being a nondemolition measurement! Detecting, e.g., which arm of a Mach-Zender interferometer a photon is in without absorbing it is extremely difficult. Often the best that can be done is determining that the photon *isn't* in one arm, by the failure of a photodetector to click.

The spin- $1/2$ can be measured by a Stern-Gerlach device. Depending on the orientation of the spin, the atom it is attached to will be deflected up or down. However, we do not know the outcome of the measurement unless we cause the atom to strike a photographic plate, at which point it is lost.

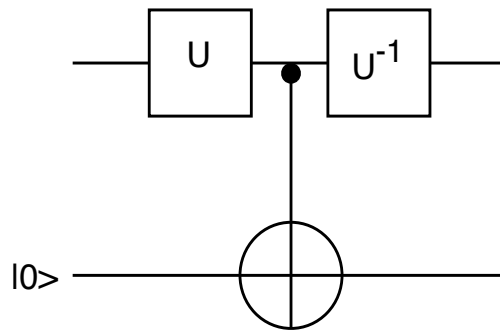
In some cases, we do not need to be able to perform ideal measurements. For many information processing protocols, it is sufficient to perform the measurements at the end. In principle, this is always possible; we can defer all measurements to the end of the procedure. This is the *principle of deferred measurement*. Let's see how it works.

Suppose we wish to measure a q-bit in the Z basis, but we still need to use it after the measurement. How can we get around this problem? Consider this quantum circuit:

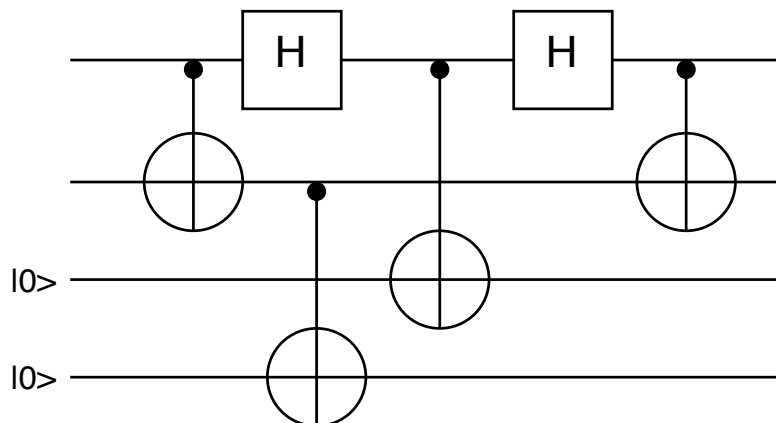


If the first bit is in the state $|0\rangle$ the second bit will be left in state $|0\rangle$; if it is in state $|1\rangle$, the second bit will be flipped. We could then measure the second bit, and we will have, indirectly, measured the first—without destroying it. Moreover, we can wait to measure the second bit until the procedure is over without altering the results.

What if we want to measure in other than the standard Z basis? Let \hat{U} change from the basis we wish to measure to the Z basis, and carry out this circuit:

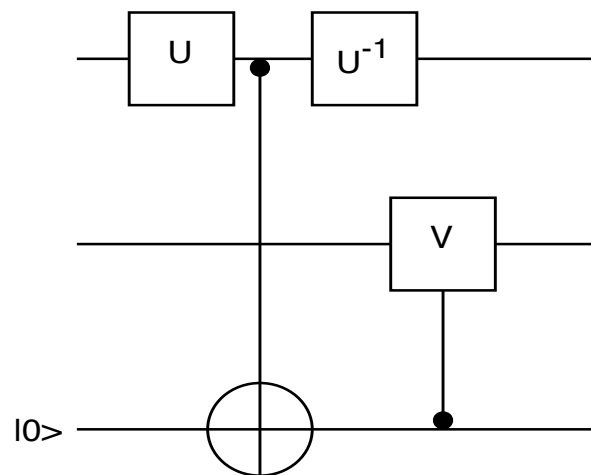


The same trick works for measurements on more than one quantum bit. Here is a circuit for a Bell state measurement:



In some protocols, we may use the outcome of a measurement to determine an action to perform. For instance, in the quantum teleportation protocol, the outcome of Alice's Bell state measurement determines which unitary transformation Bob must perform.

Even in this case, it is often possible to defer all measurements to the end of the procedure. Suppose we wish to measure a q-bit in some basis, and if the result is a 1, do a unitary transformation \hat{V} on another q-bit. We can replace this conditioned operation by a *controlled* operation:



This deferred measurement principle uses the fact that we can do measurements *indirectly*. Such indirect measurements take the following form:

1. Prepare an *extra* system (e.g., another q-bit) in a *known* initial state (e.g., $|0\rangle$). This extra system is often referred to as an *ancillary system* or *ancilla*.
2. Have the system and the ancilla interact by carrying out some circuit.
3. Measure the ancilla. This will give information about the system, thereby indirectly measuring it as well.

In practice, virtually all real measurements are indirect. Even our own eyesight works indirectly, by intercepting light which has bounced off of the object we are viewing.

Here is an example of an indirect measurement used in quantum information:

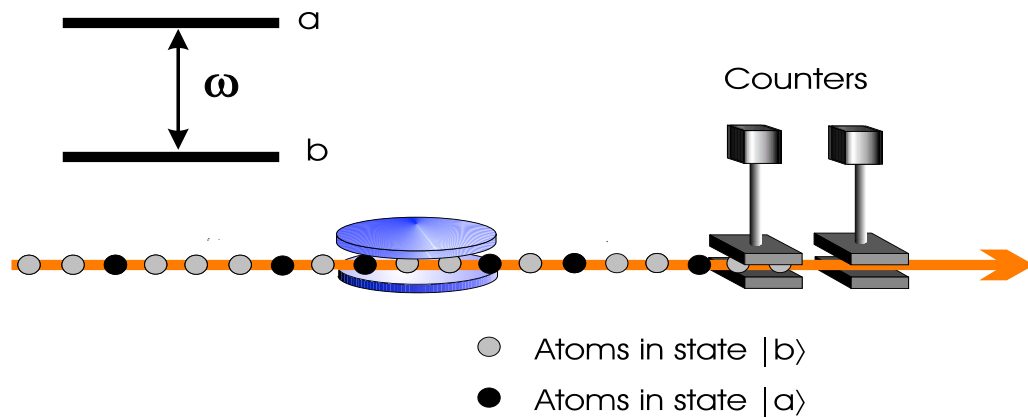


Figure 1 - Kist - PRA

The system we wish to measure is the microwave mode in the superconducting cavity. We probe it by preparing atoms in a particular state and sending them through the cavity; while inside, they interact with the microwaves, and are measured when they emerge.

In fact, the use of ancillas gives us more than just a way of doing projective measurements. It enables us to carry out a broad range of actions, of which projective measurements are just a special case. These more general procedures are sometimes called *generalized measurements* or *quantum operations*.

Next time: Generalized measurements.