

The Postulates of Quantum Mechanics

- We have reviewed the mathematics (complex linear algebra) necessary to understand quantum mechanics. We now see how the *physics* of quantum mechanics fits into this mathematical framework.
- We are really defining the *structure* of a quantum theory. All physical theories based on quantum mechanics share this common structure. Later in the course, we will see how this mathematical structure is realized for realistic systems. For now, we will use our simple example of the spin-1/2 particle to illustrate these ideas.

Postulate 1: State space

- Every physical system has associated a Hilbert space \mathcal{H} of some dimension D , known as the state space of that system; and the system is completely described by its state vector, which is a unit vector in the state space.
- If we choose a particular basis for the Hilbert space $|j\rangle$, $j = 1, \dots, D$, a state can be written in the form

$$|\psi\rangle = \sum_{j=1}^D \alpha_j |j\rangle, \quad \text{where } \sum_j |\alpha_j|^2 = 1.$$

- In the case of spin-1/2, we can write any state in terms of the basis “spin up” and “spin down” along the Z axis:

$$|\psi\rangle = \alpha_1 |\uparrow_Z\rangle + \alpha_2 |\downarrow_Z\rangle.$$

- The overall phase of the state has no physical meaning, so $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ represent the same physical state.
- The choice of basis relates to possible measurements of the system. As we will see, each basis is associated with a particular measurement (or group of compatible measurements), and each basis vector with a particular measurement outcome.
- For spin-1/2 each basis is associated with a particular direction in space along which the component of the spin could be measured. So $\{|\uparrow_z\rangle, |\downarrow_z\rangle\}$ is associated with measurement of the component of spin along the Z axis, with the basis vectors corresponding to spin up or down. Similarly, the bases $\{|\uparrow_x\rangle, |\downarrow_x\rangle\}$ and $\{|\uparrow_y\rangle, |\downarrow_y\rangle\}$ represent other possible measurements.

Postulate 2: Unitary time evolution

- The time-evolution of a closed system is described by a unitary transformation,

$$|\psi(t_2)\rangle = \hat{U}(t_2, t_1)|\psi(t_1)\rangle,$$

where \hat{U} is independent of the initial state.

- In fact, the time-evolution of the state is given by the Schrödinger equation

$$i\hbar \frac{d|\psi\rangle}{dt} = \hat{H}(t)|\psi\rangle,$$

where $\hat{H}(t)$ is an Hermitian operator (the *Hamiltonian*) and which describes the *energy* of the system. How does this relate to unitary transformations?

This is easiest to see if \hat{H} is a fixed operator (i.e., constant in time). In that case, a solution to Schrödinger's equation is

$$|\psi(t_2)\rangle = \exp(-i\hat{H}(t_2 - t_1)/\hbar)|\psi(t_1)\rangle.$$

The operator $-\hat{H}(t_2 - t_1)/\hbar$ is Hermitian, so the operator

$$\hat{U}(t_2, t_1) = \exp(-i\hat{H}(t_2 - t_1)/\hbar)$$

is unitary, as asserted.

Suppose there is a uniform magnetic field in the Z direction. Then states with spin up and down along the Z axis have different energies. This is represented by a Hamiltonian

$$\hat{H} = \begin{pmatrix} E_0 & 0 \\ 0 & -E_0 \end{pmatrix} \equiv E_0 \hat{Z},$$

where E_0 is proportional to the strength of the magnetic field. If $|\psi\rangle = \alpha|\uparrow_Z\rangle + \beta|\downarrow_Z\rangle$ at $t = 0$, then

$$|\psi(t)\rangle = \alpha e^{-iE_0 t/\hbar} |\uparrow_Z\rangle + \beta e^{iE_0 t/\hbar} |\downarrow_Z\rangle.$$

This type of evolution is equivalent to a steady rotation about the Z axis, called *precession*.

- If $|\psi\rangle$ is an eigenstate of \hat{H} , $|\uparrow_Z\rangle$ or $|\downarrow_Z\rangle$, the only effect is a change in the global phase of the state which has no physical consequences:

$$|\psi(t)\rangle = e^{-iE_0t/\hbar}|\uparrow_Z\rangle \text{ or } e^{+iE_0t/\hbar}|\downarrow_Z\rangle$$

Because of this, we call these *stationary states*.

- Similarly, we could have a uniform field in the X direction or the Y direction. In these cases, the Hamiltonians would be $E_0\hat{X}$ or $E_0\hat{Y}$, and the stationary states would be the \hat{X} or \hat{Y} eigenstates.
- If the Hamiltonian $\hat{H}(t)$ is not constant, the situation is more complicated; but the time evolution in every case is still given by a unitary transformation.

Controlling the Hamiltonian

- A common situation in quantum information is when we have some control over the Hamiltonian of the system. For instance, we could turn on a uniform magnetic field in the Z direction, leave it on for a time τ , and then turn it off. In that case, the state will have evolved by

$$|\psi\rangle \rightarrow \exp(i\theta\hat{Z})|\psi\rangle \equiv \hat{U}|\psi\rangle,$$

where $\theta = -E_0\tau/\hbar$. In this case, we say we have “performed a unitary transformation \hat{U} on the system.”

- The Hamiltonian has the time dependence $\hat{H}(t) = f(t)E_0\hat{Z}$ where $f(t) = 1$ for $0 \leq t \leq \tau$ and $f(t) = 0$ otherwise.

Postulate 3: Measurement

- An observable is a measurable quantity, which is associated with an Hermitian operator $\hat{O} = \hat{O}^\dagger$. In measuring an observable \hat{O} , the possible measurement outcomes are given by the eigenvalues λ_j ; these occur with probabilities equal to the square of the amplitude for that outcome, and the system is left in an eigenstate of \hat{O} .
- Suppose one is given a system in a state $|\psi\rangle$, and wishes to make a measurement of an observable \hat{O} . By the spectral theorem

$$\hat{O} = \sum_{j=1}^M \lambda_j \hat{\mathcal{P}}_j, \quad \sum_{j=1}^M \hat{\mathcal{P}}_j = \hat{I}, \quad \hat{\mathcal{P}}_j \hat{\mathcal{P}}_k = \delta_{jk} \hat{\mathcal{P}}_j.$$

where $M \leq D$ and the projectors $\hat{\mathcal{P}}_j$ are a decomposition of the identity.

- The outcome λ_j occurs with probability $p_j = \langle \psi | \hat{\mathcal{P}}_j | \psi \rangle = \langle \hat{\mathcal{P}}_j \rangle$, and the system is left in the (renormalized) state

$$|\psi'\rangle = \hat{\mathcal{P}}_j |\psi\rangle / \sqrt{p_j}.$$

This is called *Born's Rule*. The expectation value of the measurement outcomes is $\langle \hat{O} \rangle = \langle \psi | \hat{O} | \psi \rangle$.

- If $M = D$ so the $\hat{\mathcal{P}}_j$ are projectors $|\phi_j\rangle\langle\phi_j|$ onto eigenvectors of \hat{O} , then we can write $|\psi\rangle$ in the eigenbasis:

$$|\psi\rangle = \sum \alpha_j |\phi_j\rangle.$$

Outcome λ_j occurs with probability $p_j = |\alpha_j|^2$ and the system is afterwards left in the state $|\phi_j\rangle$.

Equivalent Observables

- Two observables $\hat{A} = \hat{A}^\dagger$ and $\hat{B} = \hat{B}^\dagger$ with the same spectral decomposition $\{\hat{\mathcal{P}}_j\}$ will have the same number of outcomes, with the same probabilities, and will leave the system in the same final states. The only difference between \hat{A} and \hat{B} is the numerical value of the eigenvalues λ_j . Such a pair of observables is *equivalent*.
- If \hat{A} and \hat{B} are equivalent, with eigenvalues $\{\lambda_j\}$ and $\{\lambda'_j\}$, we can define a function f such that $f(\lambda_j) = \lambda'_j$ for all j . In this case, $\hat{B} = f(\hat{A})$.
- Similarly, if f is any real invertible function, then \hat{A} and $f(\hat{A})$ are equivalent observables.

- Because observables with different eigenvalues but the same spectral decomposition are equivalent, one often specifies a measurement just by the decomposition of the identity $\{\hat{\mathcal{P}}_j\}$. In this case, the projectors are often called *measurement operators*.
- In this way of treating measurements, the outcomes are simply labeled by the number j rather than by the value of a physical quantity. (For example, “Spin Up” versus “Spin Down.”)

Example: Measuring Spin-1/2

The observables of a spin-1/2 are 2×2 Hermitian matrices. Any *nontrivial* observable has two distinct eigenvalues corresponding to two orthogonal eigenvectors:

$$\hat{O} = \lambda_1 |\phi_1\rangle\langle\phi_1| + \lambda_2 |\phi_2\rangle\langle\phi_2|.$$

The most general orthonormal basis for a spin-1/2 is

$$\begin{array}{l} |\phi_1\rangle \\ |\phi_2\rangle \end{array} = \begin{array}{l} \cos(\theta) \\ \sin(\theta) \end{array} |\uparrow_z\rangle \pm \begin{array}{l} \sin(\theta) \\ \cos(\theta) \end{array} e^{i\phi} |\downarrow_z\rangle.$$

Any measurement with this decomposition is equivalent to measuring the observable $\hat{O} = \vec{n} \cdot \hat{\vec{\sigma}}$, where \vec{n} is the unit vector in the direction given by θ, ϕ . \hat{O} is like one of the Pauli matrices: it has eigenvalues ± 1 and is idempotent.

Compatible Measurements

- If two observables commute, $\hat{A}\hat{B} = \hat{B}\hat{A}$, they are *compatible*. Both can be measured simultaneously, and a state can simultaneously be an eigenstate of both.
- A set \hat{A}, \hat{B}, \dots of commuting observables is *complete* if there is a *unique* orthonormal eigenbasis $\{|\phi_j\rangle\}$ for all of them simultaneously. \hat{A}, \hat{B}, \dots are simultaneously diagonalizable in the basis $|\phi_j\rangle$. A single observable \hat{A} is complete if and only if it is *nondegenerate*.
- For any orthonormal basis, there is a complete set of observables which picks that basis out. This connects a basis (which is a mathematical object) with a measurement (which is a physical procedure). The vectors in the basis correspond to possible outcomes of some complete set of measurements.

Distinguishing States

- Suppose we have a physical system, which we know has been prepared in one of two possible states $|\psi\rangle$ or $|\phi\rangle$. By measuring the system, can we tell in which state it was prepared? We would like to find an observable \hat{O} with measurement outcomes 1 and 2, so that in state $|\psi\rangle$ $p_1 = 1$ and $p_2 = 0$, and in state $|\phi\rangle$ $p_1 = 0$ and $p_2 = 1$.
- This is only possible if $|\psi\rangle$ and $|\phi\rangle$ are both eigenstates of \hat{O} . But since eigenstates of a Hermitian operator corresponding to distinct eigenvalues must be orthogonal, that requires $\langle\psi|\phi\rangle = 0$.
- If $|\psi\rangle$ and $|\phi\rangle$ are *not* orthogonal, than there is no measurement which can reliably tell them apart. In this case we say they are *indistinguishable*.

Generalized Measurements

- This description of measurement is rather idealized. In quantum information theory, it is often advantageous to allow a much wider range of possible procedures, as we will see later in the course. These procedures are called *generalized measurements*. When we wish to specify the type of measurement we have just described, we call it a *projective measurement*, or a *von Neumann measurement*. Projective measurements are a special case of generalized measurements.
- The existence of more general measurement procedures does not contradict the postulate presented here. As we shall see, it is possible to use projective measurements to realize any generalized measurement.

Postulate 4: Composite systems

- *If a composite system is composed of subsystems A and B which have associated Hilbert spaces \mathcal{H}_A and \mathcal{H}_B , then the associated Hilbert space of the joint system is the tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B$.*
- Everything we have already learned about quantum mechanics generalizes to the case where the system is part of a composite system. Let's go through them point by point.

Acting on a Subsystem

1. If subsystem A is in state $|\psi\rangle$ and subsystem B is in state $|\phi\rangle$, then the *joint system* is in the product state $|\psi\rangle \otimes |\phi\rangle$.
2. If \hat{U}_A is a unitary transformation which acts on subsystem A, then $\hat{U}_A \otimes \hat{I}$ is the corresponding unitary for the joint system. Similarly, if \hat{U}_B acts on B, then $\hat{I} \otimes \hat{U}_B$ is the unitary for the joint system.

$$(\hat{U}_A \otimes \hat{I})(|\psi\rangle \otimes |\phi\rangle) = (\hat{U}_A|\psi\rangle) \otimes |\phi\rangle.$$

$$(\hat{I} \otimes \hat{U}_B)(|\psi\rangle \otimes |\phi\rangle) = |\psi\rangle \otimes (\hat{U}_B|\phi\rangle).$$

3. If \hat{A} is an observable for subsystem A, then $\hat{A} \otimes \hat{I}$ is the corresponding observable for the joint system.

- Note that $\hat{A} \otimes \hat{I}$ and $\hat{I} \otimes \hat{B}$ *always* commute, and therefore are compatible observables. Physically, this means that measurements on different subsystems can always be done simultaneously.
- Even if \hat{A} is nondegenerate, $\hat{A} \otimes \hat{I}$ is *not*, and similarly for $\hat{I} \otimes \hat{B}$. However, if \hat{A} and \hat{B} are *both* nondegenerate, together $\hat{A} \otimes \hat{I}$ and $\hat{I} \otimes \hat{B}$ are a complete set of observables.
- More generally, a complete set of observables on A together with a complete set of observables on B is a complete set of observables on the composite system.

Treating Subsystems Jointly

- While we can extend the results for single systems to composite systems, there are many more possible states, evolutions and measurements that are allowed.
- Most states $|\Psi\rangle$ of a composite system are *not* product states. For an example with two spin-1/2 systems,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle$$

is not a product state. For this joint state, we cannot assign well-defined states to the subsystems. Such a joint state is called *entangled*.

- A consequence of entanglement is that measurements on the subsystems will in general be *correlated*.

Interactions

- We can also perform joint unitaries \hat{U} on both systems at once. If the initial state is a product $|\psi\rangle \otimes |\phi\rangle$, then after applying a joint unitary the system will in general be entangled. Correlations are produced by *interaction* between the spins.
- For example, we might have a Hamiltonian of the form

$$\hat{H} = E_0 \hat{Z} \otimes \hat{Z}.$$

The unitary operators $\exp(i\theta\hat{H})$ produced by this Hamiltonian will *not* be product operators, even though \hat{H} itself *is* a product operator. So initial product states will evolve to become entangled.

Joint Measurements

- Finally, we can measure observables \hat{O} which are *not* product operators. The measurement process follows the same rules we have already seen: an eigenvalue of \hat{O} will occur with some probability, and the system will be left in the corresponding eigenstate of \hat{O} . However, since \hat{O} is not a product operator, the eigenstates of \hat{O} will in general be *entangled* states.
- This means that entanglement can be produced by joint measurements even if the initial state $|\psi\rangle \otimes |\phi\rangle$ was a product.

Quantum Information Processing

From this mathematical description of quantum mechanics, we see what we can use to build information processing protocols:

1. We can prepare quantum systems in particular states.
2. We can do a series of unitary transformations and measurements on the systems (where later operations may depend on the results of earlier measurements).
3. The output of the process will be the result of some final measurement or measurements.

We will see in this course how, from these building blocks, we can construct information processing protocols more powerful than any classical system.