

Complex Linear Algebra

- The basic mathematical objects in quantum mechanics are *state vectors* and *linear operators* (matrices). Because the theory is fundamentally linear, and the probability amplitudes are complex numbers, the mathematics underlying quantum mechanics is complex linear algebra. The vectors are members of a complex vector space, or *Hilbert* space, with an associated inner product.
- It is important to remember that these abstract mathematical objects represent physical things, and should be considered independent of the particular *representations* they are given by choosing a particular basis.

Dirac notation

- State vectors in quantum mechanics are written in *Dirac notation*. The basic object is the *ket-vector* $|\psi\rangle$, which (given a particular basis) can be represented as a *column vector*. The adjoint of a ket-vector is a *bra-vector* $\langle\psi|$, represented as a *row vector*.

$$|\psi\rangle = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix}, \quad \langle\psi| = (\alpha_1^* \cdots \alpha_N^*).$$

- If the vector $|\psi\rangle$ is normalized, that means

$$\sum_{j=1}^N |\alpha_j|^2 = 1.$$

Inner and Outer Products

Given two vectors $|\psi\rangle$ and $|\phi\rangle$,

$$|\psi\rangle = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix}, \quad |\phi\rangle = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix},$$

the inner product between them is written

$$\langle\phi|\psi\rangle = (\beta_1^* \cdots \beta_N^*) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} = \sum_j \beta_j^* \alpha_j.$$

The inner product is independent of the choice of basis.

- $\langle \phi | \psi \rangle$ is called a *bracket*. Note that $\langle \phi | \psi \rangle = \langle \psi | \phi \rangle^*$, and for a normalized vector $\langle \psi | \psi \rangle = 1$. If two vectors are orthogonal then $\langle \psi | \phi \rangle = 0$.
- It is also handy to write the *outer product* (also sometimes called a *dyad*):

$$|\psi\rangle\langle\phi| = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} (\beta_1^* \cdots \beta_N^*) = \begin{pmatrix} \alpha_1\beta_1^* & \cdots & \alpha_1\beta_N^* \\ \vdots & \ddots & \vdots \\ \alpha_N\beta_1^* & \cdots & \alpha_N\beta_N^* \end{pmatrix}.$$

- A dyad $|\psi\rangle\langle\phi|$ is a *linear operator*. As we shall see, it is common (and often convenient) to write more general operators as linear combinations of dyads.

Linear operators

A *linear operator* \hat{O} transforms states to states such that

$$\hat{O}(a|\psi\rangle + b|\phi\rangle) = a\hat{O}|\psi\rangle + b\hat{O}|\phi\rangle$$

for all states $|\psi\rangle, |\phi\rangle$ and complex numbers a, b . Given a choice of basis, an operator can be represented by a *matrix*

$$\hat{O} = \begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{pmatrix} \equiv [a_{ij}].$$

The matrix representation depends on the choice of basis. We will only be dealing with orthonormal bases in this class.

Expectation values

Given a state $|\psi\rangle$ and an operator \hat{O} , we can calculate a number

$$\langle \hat{O} \rangle \equiv \langle \psi | \hat{O} | \psi \rangle = \langle \psi | \left(\hat{O} | \psi \rangle \right)$$

which is called *the expectation value of \hat{O} in the state $|\psi\rangle$* . Given a particular choice of basis, we can express this number in terms of the elements of \hat{O} and $|\psi\rangle$:

$$\langle \hat{O} \rangle = \sum_{i,j} \alpha_i^* a_{ij} \alpha_j.$$

As we will see, when \hat{O} is Hermitian, its expectation value gives the average result of some measurement on a system in the state $|\psi\rangle$.

Matrix elements

Similar to an expectation value is a *matrix element* $\langle \psi | \hat{O} | \phi \rangle$ where $|\psi\rangle \neq |\phi\rangle$. If $|\psi\rangle$ and $|\phi\rangle$ are both members of the orthonormal basis $\{|j\rangle\}$ then

$$\langle i | \hat{O} | j \rangle = a_{ij},$$

where a_{ij} is an element of the matrix representing \hat{O} in the basis $\{|j\rangle\}$. The operator can be written as a sum over outer products,

$$\hat{O} = \sum_{ij} a_{ij} |i\rangle \langle j|.$$

Hermitian Conjugation

- One of the most important operations in complex linear algebra is *Hermitian conjugation*. The Hermitian conjugate \hat{O}^\dagger is the *complex conjugate* of the *transpose* of an operator \hat{O} . If in a particular basis $\hat{O} = [a_{ij}]$ then $\hat{O}^\dagger = [a_{ji}^*]$.
- Hermitian conjugation works similarly to transposition in real linear algebra: $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$. When applied to state vectors, $(|\psi\rangle)^\dagger = \langle\psi|$. Similarly, for dyads $(|\psi\rangle\langle\phi|)^\dagger = |\phi\rangle\langle\psi|$.
- Note that Hermitian conjugation is *not* linear, but rather is *antilinear*.

$$(a\hat{O})^\dagger = a^*\hat{O}^\dagger, \quad (a|\psi\rangle)^\dagger = a^*\langle\psi|.$$

Orthonormal bases and the trace

It is normally most convenient to choose a particular orthonormal basis $\{|j\rangle\}$ and work in terms of it. As long as one works within a fixed basis, one can treat state vectors as column vectors and operators as matrices. An orthonormal basis for an N -dimensional space has N vectors that satisfy

$$\langle i|j\rangle = \delta_{ij}, \quad \sum_{j=1}^N |j\rangle\langle j| = \hat{I}.$$

The *trace* of an operator is the sum of the diagonal elements:

$$\text{Tr}\{\hat{O}\} = \sum_j \langle j|\hat{O}|j\rangle = \sum_j a_{jj}.$$

A *traceless* operator has $\text{Tr}\{\hat{O}\} = 0$.

- The trace is *independent of the choice of basis*. If $\{|j\rangle\}$ and $\{|\phi_k\rangle\}$ are both orthonormal bases, then

$$\text{Tr}\{\hat{O}\} = \sum_j \langle j|\hat{O}|j\rangle = \sum_k \langle \phi_k|\hat{O}|\phi_k\rangle.$$

- The trace also has the useful *cyclic property*

$$\text{Tr}\{\hat{A}\hat{B}\} = \text{Tr}\{\hat{B}\hat{A}\}.$$

This applies to products of any number of operators:

$$\text{Tr}\{\hat{A}\hat{B}\hat{C}\} = \text{Tr}\{\hat{C}\hat{A}\hat{B}\} = \text{Tr}\{\hat{B}\hat{C}\hat{A}\}.$$

This invariance implies that $\text{Tr}\{|\phi\rangle\langle\psi|\} = \langle\psi|\phi\rangle$.

Normal operators

- A *normal operator* satisfies $\hat{O}^\dagger \hat{O} = \hat{O} \hat{O}^\dagger$. Operators are *diagonalizable* if and only if they are normal. That is, for normal \hat{O} we can always find an orthonormal basis $\{|\phi_j\rangle\}$ such that

$$\hat{O} = \sum_j \lambda_j |\phi_j\rangle \langle \phi_j|, \quad \text{Tr}\{\hat{O}\} = \sum_j \lambda_j,$$

and any diagonalizable operator must be normal.

- These values λ_j are the *eigenvalues* of \hat{O} and $\{|\phi_j\rangle\}$ the corresponding eigenvectors, $\hat{O}|\phi_j\rangle = \lambda_j|\phi_j\rangle$. If \hat{O} is *nondegenerate*—i.e., all the λ_j are distinct—then the eigenvectors are unique (up to a phase). Otherwise there is some freedom in choosing this *eigenbasis*.

Hermitian operators

- One very useful class of operators are the *Hermitian operators*: \hat{H} which satisfy $\hat{H} = \hat{H}^\dagger$. These are the complex analogue of *symmetric* matrices. They are obviously normal: $\hat{H}^\dagger \hat{H} = \hat{H}^2 = \hat{H} \hat{H}^\dagger$. The eigenvalues of a Hermitian matrix are always *real*. We will look at this in more detail later.
- We have already seen an example: the Pauli matrices.

$$\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These matrices are obviously Hermitian. It is easy to see that any 2×2 Hermitian matrix can be written $a\hat{I} + b\hat{X} + c\hat{Y} + d\hat{Z}$ for some *real* values a, b, c, d .

The Commutator

- Matrix multiplication is *noncommutative*, in general. That is, in general $\hat{A}\hat{B} \neq \hat{B}\hat{A}$. Given two operators \hat{A} and \hat{B} , their *commutator* is $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$. $[\hat{A}, \hat{B}] = 0$ if and only if \hat{A} and \hat{B} commute.
- Occasionally, one will encounter matrices that *anticommute*: $\hat{A}\hat{B} = -\hat{B}\hat{A}$. For example, the Pauli matrices anticommute with each other. In these cases, it is sometimes helpful to define the *anticommutator*:

$$\{\hat{A}, \hat{B}\} \equiv \hat{A}\hat{B} + \hat{B}\hat{A}.$$

- If two normal operators \hat{A} and \hat{B} commute, it is possible to find an eigenbasis which simultaneously diagonalizes *both* of them. (The converse is also true.)

Orthogonal projectors

- An *orthogonal projector* $\hat{\mathcal{P}}$ is an Hermitian operator that obeys $\hat{\mathcal{P}}^2 = \hat{\mathcal{P}}$. All the eigenvalues of $\hat{\mathcal{P}}$ are either 0 or 1. The *complement* of a projector $\hat{I} - \hat{\mathcal{P}}$ is *also* a projector. Note that $|\psi\rangle\langle\psi|$ is a projector for *any* normalized state vector $|\psi\rangle$. Such a projector is called *one-dimensional*; a projector more generally has dimension $d = \text{Tr}\{\hat{\mathcal{P}}\}$.
- In dimension 2, *any* projector can be written in the form

$$\hat{\mathcal{P}} = \left(\hat{I} + \vec{n} \cdot \hat{\vec{\sigma}} \right) / 2 = |\psi_{\vec{n}}\rangle\langle\psi_{\vec{n}}|,$$

where \vec{n} is a (real) unit three-vector (n_x, n_y, n_z) (i.e., with $n_x^2 + n_y^2 + n_z^2 = 1$) and $\hat{\vec{\sigma}} = (\hat{X}, \hat{Y}, \hat{Z})$. In the Bloch sphere picture, \vec{n} is the direction in space, and $\hat{\mathcal{P}}$ is the projector onto the state $|\psi_{\vec{n}}\rangle$ that is spin up along that axis.

The Spectral Theorem

- Suppose \hat{O} is a normal operator, and λ_j are its *distinct* eigenvalues (i.e., not including any repetition) for $j = 1, \dots, M \leq N$. There is a unique set of orthogonal projectors $\hat{\mathcal{P}}_j$ such that

$$\hat{O} = \sum_{j=1}^M \lambda_j \hat{\mathcal{P}}_j,$$

and $\hat{\mathcal{P}}_j \hat{\mathcal{P}}_k = \delta_{jk} \hat{\mathcal{P}}_j$, $\sum_j \hat{\mathcal{P}}_j = \hat{I}$. Such a set of projectors is called a *decomposition of the identity*. For a Hermitian operator, the eigenvalues λ_j will all be real.

- If \hat{O} has N distinct eigenvalues (is *nondegenerate*) then $\hat{\mathcal{P}}_j = |\phi_j\rangle\langle\phi_j|$ where $|\phi_j\rangle$ is an eigenvector of \hat{O} .

Unitary Operators

- A *unitary* operator satisfies $\hat{U}^\dagger \hat{U} = \hat{U} \hat{U}^\dagger = \hat{I}$. It is clearly a normal operator. All of its eigenvalues have unit norm; that is, $|\lambda_j| = 1$ for all j . This means that

$$\lambda_j = \exp(i\theta_j)$$

for real $0 \leq \theta_j < 2\pi$.

- There is a one-to-one correspondence between the Hermitian and unitary operators; for every unitary operator \hat{U} there is an Hermitian operator \hat{H} such that

$$\hat{U} = \exp(i\hat{H}).$$

(We will clarify what this means shortly.)

- We have already seen that a unitary operator is equivalent to a change of basis. This is easy to check: if $\{|j\rangle\}$ is an orthonormal basis, then so is $\{\hat{U}|j\rangle\}$:

$$(\hat{U}|i\rangle)^\dagger(\hat{U}|j\rangle) = \langle i|\hat{U}^\dagger\hat{U}|j\rangle = \langle i|j\rangle = \delta_{ij}.$$

- For the spin-1/2, the most general 2×2 unitary can be written (up to a global phase)

$$\hat{U} = \cos(\theta/2)\hat{I} + i \sin(\theta/2)\vec{n} \cdot \hat{\vec{\sigma}},$$

where \vec{n} is again a real unit three-vector (n_x, n_y, n_z) and $\hat{\vec{\sigma}} = (\hat{X}, \hat{Y}, \hat{Z})$. In the Bloch sphere picture, \hat{U} is a rotation by θ about the axis \vec{n} .

Operator space

- The space of all operators on a particular Hilbert space of dimension N is itself a Hilbert space of dimension N^2 ; sometimes this fact can be very useful. If \hat{A} and \hat{B} are operators, so is $a\hat{A} + b\hat{B}$ for any complex a, b .
- One can define an inner product on operator space. The most commonly used one is $(\hat{A}, \hat{B}) \equiv \text{Tr}\{\hat{A}^\dagger \hat{B}\}$. It is easy to see that $(\hat{A}, \hat{B}) = (\hat{B}, \hat{A})^*$, and $(\hat{A}, \hat{A}) \geq 0$ with equality only for $\hat{A} = 0$. With relation to this inner product, the Pauli matrices together with the identity form an orthogonal basis for all operators on 2-d Hilbert space.
- A linear transformation on operator space is often referred to as a *superoperator*.

Functions of Operators

It is common to write a function of an operator $f(\hat{O})$ (where f is ordinarily a function on the complex numbers) which is itself an operator. Usually $f(x)$ is defined by a Taylor series: $f(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \dots$. For the operator version, we write

$$f(\hat{O}) = c_0\hat{I} + c_1(\hat{O} - x_0\hat{I}) + c_2(\hat{O} - x_0\hat{I})^2 + \dots$$

For particular functions and operators, this series can sometimes be summed explicitly. For instance, for a *nilpotent* operator $\hat{O}^2 = 0$, the series obviously truncates after the first-order term (taking $x_0 = 0$). For *projectors*, $\hat{P}^n = \hat{P}$ for all $n \geq 1$. For *idempotent* operators obeying $\hat{O}^2 = \hat{I}$ (such as the Pauli matrices), one can sum the even and odd terms separately.

If \hat{O} is normal, we simplify by writing \hat{O} in diagonal form:

$$\hat{O} = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j|, \quad f(\hat{O}) = \sum_j f(\lambda_j) |\phi_j\rangle\langle\phi_j|.$$

The most commonly-used function is the exponential

$$\exp(x) = 1 + x + x^2/2! + \dots,$$

but others also occur from time to time:

$$\cos(x) = 1 - x^2/2 + x^4/4! - \dots$$

$$\sin(x) = x - x^3/3! + x^5/5! - \dots$$

$$\log(1 + x) = x - x^2/2 + x^3/3 - \dots$$

Polar Decomposition

There are certain special forms in which operators can always be written. One of these is the *polar decomposition*. For any linear operator \hat{O} , there is a unitary operator \hat{U} and *positive* operators \hat{A} and \hat{B} such that

$$\hat{O} = \hat{U}\hat{A} = \hat{B}\hat{U},$$

which we call the *left* and *right polar decompositions* of \hat{O} . A positive operator is an Hermitian operator with all nonnegative eigenvalues. In this case, the positive operators \hat{A} and \hat{B} are uniquely given by

$$\hat{A} \equiv \sqrt{\hat{O}^\dagger \hat{O}}, \quad \hat{B} \equiv \sqrt{\hat{O} \hat{O}^\dagger}.$$

If \hat{O} is invertible then \hat{U} is also unique, otherwise not.

Singular Value Decomposition

For any operator \hat{O} we can find unitary operators \hat{U} and \hat{V} and a diagonal real matrix \hat{D} with all nonnegative entries, such that

$$\hat{O} = \hat{U} \hat{D} \hat{V}.$$

The operator \hat{D} is unique up to a permutation of the order of the diagonal entries. If the diagonal elements of \hat{D} are all nonzero, then \hat{O} is invertible.

Tensor products

The tensor (or Kronecker) product is a way of combining two Hilbert spaces to produce a higher dimensional space. Let $|\psi\rangle$ be a state in a D_1 -dimensional Hilbert space \mathcal{H}_1 and $|\phi\rangle$ be a state in a D_2 -dimensional Hilbert space \mathcal{H}_2 . Then we define $|\psi\rangle \otimes |\phi\rangle$ to be a state in the $D_1 D_2$ -dimensional space $\mathcal{H}_1 \otimes \mathcal{H}_2$. Such a state is called a *product state*. Any state in this larger space can be written as a linear combination of product states

$$|\Psi\rangle = \sum_{\ell} \alpha_{\ell} |\psi_{\ell}\rangle \otimes |\phi_{\ell}\rangle$$

where $|\psi_{\ell}\rangle \in \mathcal{H}_1$ and $|\phi_{\ell}\rangle \in \mathcal{H}_2$.

What are the properties of this product?

$$(a|\psi\rangle + b|\psi'\rangle) \otimes |\phi\rangle = a|\psi\rangle \otimes |\phi\rangle + b|\psi'\rangle \otimes |\phi\rangle.$$

$$|\psi\rangle \otimes (a|\phi\rangle + b|\phi'\rangle) = a|\psi\rangle \otimes |\phi\rangle + b|\psi\rangle \otimes |\phi'\rangle.$$

We need also to define bra-vectors, and the inner product:

$$(|\psi\rangle \otimes |\phi\rangle)^\dagger = \langle\psi| \otimes \langle\phi|.$$

$$(\langle\psi'| \otimes \langle\phi'|)(|\psi\rangle \otimes |\phi\rangle) = \langle\psi'|\psi\rangle\langle\phi'|\phi\rangle.$$

If $\{|j\rangle_1\}$ is a basis for \mathcal{H}_1 and $\{|k\rangle_2\}$ is a basis for \mathcal{H}_2 then $\{|j\rangle_1 \otimes |k\rangle_2\}$ is a basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$. Given two states in \mathcal{H}_1 and \mathcal{H}_2

$$|\psi\rangle = \sum_{j=1}^{D_1} \alpha_j |j\rangle_1, \quad |\phi\rangle = \sum_{k=1}^{D_2} \beta_k |k\rangle_2,$$

in terms of this basis

$$|\psi\rangle \otimes |\phi\rangle = \sum_{j,k} \alpha_j \beta_k |j\rangle_1 \otimes |k\rangle_2.$$

Generic states in $\mathcal{H}_1 \otimes \mathcal{H}_2$ are *not* product states:

$$|\Psi\rangle = \sum_{j,k} t_{jk} |j\rangle_1 \otimes |k\rangle_2.$$

Operator Tensor Products

If \hat{A} is an operator on \mathcal{H}_1 and \hat{B} on \mathcal{H}_2 , we construct a similar product to get a new operator $\hat{A} \otimes \hat{B}$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Its properties are similar to tensor products of states:

$$(a\hat{A} + b\hat{A}') \otimes \hat{B} = a\hat{A} \otimes \hat{B} + b\hat{A}' \otimes \hat{B}.$$

$$\hat{A} \otimes (a\hat{B} + b\hat{B}') = a\hat{A} \otimes \hat{B} + b\hat{A} \otimes \hat{B}'.$$

$$(\hat{A} \otimes \hat{B})^\dagger = \hat{A}^\dagger \otimes \hat{B}^\dagger.$$

$$(\hat{A} \otimes \hat{B})(\hat{A}' \otimes \hat{B}') = \hat{A}\hat{A}' \otimes \hat{B}\hat{B}'.$$

$$\text{Tr}\{\hat{A} \otimes \hat{B}\} = \text{Tr}\{\hat{A}\}\text{Tr}\{\hat{B}\}.$$

We can also apply these tensor product operators to tensor product states:

$$(\hat{A} \otimes \hat{B})(|\psi\rangle \otimes |\phi\rangle) = \hat{A}|\psi\rangle \otimes \hat{B}|\phi\rangle.$$

$$(\langle\psi| \otimes \langle\phi|)(\hat{A} \otimes \hat{B}) = \langle\psi|\hat{A} \otimes \langle\phi|\hat{B}.$$

A general operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is *not* a product operator, but can be written as a linear combination of product operators:

$$\hat{O} = \sum_{\ell} \hat{A}_{\ell} \otimes \hat{B}_{\ell}.$$

Tensor products play an important role in quantum mechanics! They describe how the Hilbert spaces of subsystems are combined.

Matrix representation

What does the matrix representation of a tensor product look like? If $|\psi\rangle$ has amplitudes $(\alpha_1, \dots, \alpha_{D_1})$ and $|\phi\rangle$ has amplitudes $(\beta_1, \dots, \beta_{D_2})$, the state $|\psi\rangle \otimes |\phi\rangle$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ is represented as a $D_1 D_2$ -dimensional column vector:

$$|\psi\rangle \otimes |\phi\rangle = \begin{pmatrix} \alpha_1 |\phi\rangle \\ \alpha_2 |\phi\rangle \\ \vdots \\ \alpha_{D_1} |\phi\rangle \end{pmatrix}.$$

Similarly, if $\hat{A} = [a_{ij}]$ and $\hat{B} = [b_{ij}]$ then

$$\hat{A} \otimes \hat{B} = \begin{pmatrix} a_{11}\hat{B} & \cdots & a_{1D_1}\hat{B} \\ \vdots & \ddots & \vdots \\ a_{D_11}\hat{B} & \cdots & a_{D_1D_1}\hat{B} \end{pmatrix}.$$

For brevity, $|\psi\rangle \otimes |\phi\rangle$ is often written $|\psi\rangle|\phi\rangle$ or even $|\psi\phi\rangle$. One can similarly condense the notation for operators; but one should be very careful not to confuse $\hat{A} \otimes \hat{B}$ with $\hat{A}\hat{B}$.

Example: Two Spins

Given two states in the Z basis, $|\psi\rangle = \alpha_1|\uparrow\rangle + \alpha_2|\downarrow\rangle$ and $|\phi\rangle = \beta_1|\uparrow\rangle + \beta_2|\downarrow\rangle$, we can write

$$|\psi\rangle \otimes |\phi\rangle = \alpha_1\beta_1|\uparrow\uparrow\rangle + \alpha_1\beta_2|\uparrow\downarrow\rangle + \alpha_2\beta_1|\downarrow\uparrow\rangle + \alpha_2\beta_2|\downarrow\downarrow\rangle.$$

A general state of two spins would be

$$|\Psi\rangle = t_{11}|\uparrow\uparrow\rangle + t_{12}|\uparrow\downarrow\rangle + t_{21}|\downarrow\uparrow\rangle + t_{22}|\downarrow\downarrow\rangle.$$

As a column vector this is

$$|\Psi\rangle = \begin{pmatrix} t_{11} \\ t_{12} \\ t_{21} \\ t_{22} \end{pmatrix}.$$

Similarly, operators on two spins can be written as linear combinations of tensor products of the Pauli matrices and the identity. For instance,

$$\hat{I} \otimes \hat{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \hat{Z} \otimes \hat{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\hat{I} \otimes \hat{X} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \hat{X} \otimes \hat{Y} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$