

Homework

- (1) Show that the induced topology and the quotient topology fulfill the axioms of a topological space.
- (2) Prove that $S^1 = \{x^2 + y^2 = 1\}$ with the induced topology (from \mathbf{R}^2) is homeomorphic to $S^1 = [0, 1]/\sim$ with the quotient topology.
- (3) Prove that S^1 (with either topology in Problem (2)) is a topological manifold.
- (4) Give an example of a Hausdorff, second countable topological space which is not a topological manifold.
- (5) Let $\phi : V \rightarrow W$ be a linear map between vector spaces (over a fixed ground field k). Prove the following:
 - (a) $\text{Ker}(\phi)$ is a vector subspace of V .
 - (b) $\text{Im}(\phi)$ is a vector subspace of W .
- (6) Let $V \subset W$ be a vector subspace. Prove that the quotient W/V is a vector space.
- (7) Let V^* (called the *dual of V*) be the set of linear maps from V to the base field k . Prove that V^* is a vector space over k . Assuming V is finite-dimensional, exhibit a basis for V^* in terms of a basis for V .
- (8) Prove that $S^n = \{x_0^2 + \dots + x_n^2 = 1\} \subset \mathbf{R}^{n+1}$ is a smooth n -dimensional manifold, by taking stereographic projections.
- (9) Complete the proof that \mathbf{RP}^n is a smooth n -dimensional manifold.
- (10) Define $\mathbf{CP}^n = (\mathbf{C}^{n+1} - \{(0, \dots, 0)\})/\sim$, where $(z_0, \dots, z_n) \sim (tz_0, \dots, tz_n)$, $t \in \mathbf{C} - \{0\}$. Prove that \mathbf{CP}^n is a smooth $2n$ -dimensional manifold. (Recall that $\mathbf{C} \xrightarrow{\sim} \mathbf{R}^2$, where $z = x + iy \mapsto (x, y)$.)
- (11) Complete the proof that $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ is a smooth manifold of dimension 2.
- (12) Prove that $S^n = \{x_0^2 + \dots + x_n^2 = 1\} \subset \mathbf{R}^{n+1}$ is an n -dimensional manifold by showing it is a regular value of some map.
- (13) Prove that if M, N are manifolds, $f : M \rightarrow N$ is a submersion, and $U \subset M$ is open, then $f(U)$ is open in N .
- (14) Prove that if M is compact and N is connected, every submersion $f : M \rightarrow N$ is surjective. Prove that there is no submersion from a compact manifold to \mathbf{R}^n .
- (15) Define $SL(2, \mathbf{R}) = \{A \in M_2 \mid \det(A) = 1\}$, where M_n is the set of n by n matrices with real entries. Prove that $SL(2, \mathbf{R})$ is a smooth manifold. What is its dimension? ($SL(n, \mathbf{R})$ is called the *special linear group* over \mathbf{R} of dimension n .)
- (16) Let N be an n -dimensional submanifold of M (of dimension m) and $x \in N$. Prove that there exists a local coordinate system $\{x_1, \dots, x_m\}$ for M on $U \ni x$ so that $U \cap N = \{x_{n+1} = 0, \dots, x_m = 0\}$.
- (17) Define the *orthogonal group* $O(n) = \{A \in M_n \mid AA^T = I\}$, where A^T is the transpose of A . Prove that $O(n)$ is a manifold as follows: Consider the map $\phi : M_n \rightarrow \text{Sym}(n)$ which sends $A \mapsto AA^T$, where $\text{Sym}(n)$ is the set of symmetric $n \times n$ matrices (i.e., $B^T = B$).

- (a) Show that $\text{Sym}(n)$ is a manifold. Compute its dimension.
 (b) Compute the derivative of ϕ and show ϕ is a submersion.
 (c) What is the dimension of $O(n)$?
 (d) Prove that $O(n)$ is compact.
- (18) Let $F(x_1, \dots, x_n)$ be a *homogeneous* polynomial of degree d in n (real) variables, i.e.,

$$F(tx_1, \dots, tx_n) = t^d \cdot F(x_1, \dots, x_n).$$

- (a) Prove Euler's identity:

$$\sum_{i=1}^n x_i \cdot \frac{\partial F}{\partial x_i} = d \cdot F.$$

- (b) Prove that the set $F^{-1}(a)$, $a \neq 0$, is a submanifold of \mathbf{R}^n .
- (19) Give a detailed proof of the equivalence of the three definitions of $T_p M$ given in class. Pay special attention to good exposition.
- (20) Recall that \mathcal{F}_p is the set of germs of functions on a manifold M which vanish at $p \in M$. Let \mathcal{F}_p^k be the ideal of $C^\infty(p)$ generated by $f_1 \cdots f_k$, where $f_i \in \mathcal{F}_p$. (This means that every element of \mathcal{F}_p^k is a sum $\sum_i g_i f_{i1} \cdots f_{ik}$, $g_i \in C^\infty(p)$, $f_{ij} \in \mathcal{F}_p$.)
- (a) Prove that, in every coordinate system (x_1, \dots, x_n) , an element $f \in \mathcal{F}_p^k$ has a Taylor expansion which vanishes up to order k .
 (b) Compute the dimension of $\mathcal{F}_p^k / \mathcal{F}_p^{k+1}$.
 (c) Construct a smooth manifold $E \xrightarrow{\pi} M$ whose fiber at $p \in M$ is $\mathcal{F}_p^1 / \mathcal{F}_p^3$. (This involves writing down coordinate charts and computing transition functions.)
- (21) Consider the cotangent bundle $\pi : T^*M \rightarrow M$. In class we gave an atlas for T^*M in terms of $\pi^{-1}(U_\alpha)$, where $\{U_\alpha\}$ was an atlas for M . Compute the Jacobian for the transition functions on the overlaps $\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)$.
- (22) Prove that $d(fg) = fdg + gdf$.
- (23) Consider the map $i : S^1 = [0, 2\pi]/(0 \sim 2\pi) \rightarrow \mathbf{R}^2$, $\theta \mapsto (\cos \theta, \sin \theta)$. Compute $i^*((x^2 + y)dx + (3 + xy^2)dy)$.
- (24) In class we defined the *derivative map* as follows: Let $\phi : M \rightarrow N$ be a smooth map between manifolds. Then the derivative $\phi_* : T_p M \rightarrow T_{\phi(p)} N$ is given by $X \mapsto X \circ \phi^*$, where $X : C^\infty(p) \rightarrow \mathbf{R}$ is a derivation and ϕ^* is the pullback $C^\infty(\phi(p)) \rightarrow C^\infty(p)$. Give an equivalent definition for ϕ_* in terms of Definition 1 of a tangent space.
- (25) Let $\phi : M \rightarrow N$ be a smooth map between manifolds. Prove that the following diagram commutes:

$$\begin{array}{ccc} \Omega^0(N) & \xrightarrow{\phi^*} & \Omega^0(M) \\ d \downarrow & \circlearrowleft & \downarrow d \\ \Omega^1(N) & \xrightarrow{\phi^*} & \Omega^1(M) \end{array}$$

- (26) Let $\phi : [a, b] \rightarrow [c, d]$ be a diffeomorphism with coordinates s for $[a, b]$ and t for $[c, d]$. A global 1-form ω on $[c, d]$ can be written as $f(t)dt$, for some smooth function $f(t)$.
- (a) Write $\phi^*\omega$ in terms of coordinates s on $[a, b]$.

- (b) Now define the *integral* of ω on $[c, d]$ to be $\int_{[c,d]} \omega = \int_c^d f(t)dt$. Similarly define $\int_{[a,b]} \phi^* \omega$. Prove that

$$\int_{[c,d]} \omega = \int_{[a,b]} \phi^* \omega.$$

- (27) Prove that $SO(2)$ consists of the 2×2 matrices $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$, where $\theta \in \mathbf{R}$.

Show that $SO(2)$ is diffeomorphic to S^1 .

- (28) Explain why $SO(n)$ is a Lie group. Explain why $GL(n, \mathbf{C})$ is a Lie group.
- (29) Show that $GL(n, \mathbf{C}) \subset GL^+(2n, \mathbf{R})$.
- (30) Let $f : M \rightarrow N$ and $g : L \rightarrow M$ be smooth maps between smooth manifolds, and let ω be a 1-form on N . Then prove that $(f \circ g)^* \omega = g^* \circ (f^* \omega)$.
- (31) Let V, W be finite-dimensional \mathbf{R} -vector spaces.
- Prove that $Hom(V, W)$ is an \mathbf{R} -vector space.
 - Show that $\dim(Hom(V, W)) = \dim V \cdot \dim W$.
 - Find a natural linear map $V^* \otimes W \rightarrow Hom(V, W)$ and prove that it is an isomorphism. It follows that $V \otimes W$ has dimension $\dim V \cdot \dim W$. (Here, a *natural isomorphism* is an isomorphism which does not depend on a choice of basis.)
- (32) Let V, W , and U be \mathbf{R} -vector spaces. Prove that $V \otimes W$ is naturally isomorphic to $W \otimes V$ and that $(V \otimes W) \otimes U$ is naturally isomorphic to $V \otimes (W \otimes U)$.
- (33) Let V be a 2-dimensional vector space with basis $\{v_1, v_2\}$ and $A : V \rightarrow V$ be a linear map given by $v_1 \mapsto 5v_1 + 6v_2$, $v_2 \mapsto 3v_1 + 2v_2$. Then write a matrix for $A \otimes A : V \otimes V \rightarrow V \otimes V$ in terms of the basis $\{v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2\}$.
- (34) Let V be a vector space of dimension n . Show that $\bigwedge^n V \simeq \mathbf{R}$ as follows:
- Find an alternating multilinear form $\phi : V \times \cdots \times V \rightarrow \mathbf{R}$, (n copies of V). (Prove it's alternating.)
 - Explain how to use the universal property to show $\bigwedge^n V \simeq \mathbf{R}$.
- Note that ϕ is the determinant map, up to a normalization.
- (35) Let V be a vector space. Show there exists a (well-defined) linear map $\bigwedge^k V \otimes \bigwedge^l V \rightarrow \bigwedge^{k+l} V$ which sends $(v_1 \wedge \cdots \wedge v_k) \otimes (w_1 \wedge \cdots \wedge w_l) \mapsto v_1 \wedge \cdots \wedge v_k \wedge w_1 \wedge \cdots \wedge w_l$.
- (36) If V is a finite-dimensional vector space of dimension n , then show there exists a natural isomorphism $\bigwedge^{n-k} V \simeq (\bigwedge^k V)^*$.
- (37) Let M be a manifold. Prove that d satisfies the formula $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta$, where $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$.
- (38) Let $\phi : M \rightarrow N$ be a smooth map between manifolds and $\omega \in \Omega^k(N)$. With respect to local coordinates x_1, \dots, x_m for M and y_1, \dots, y_n for N , we defined

$$\phi^* \omega = \sum_{i_1, \dots, i_k} f_{i_1, \dots, i_k}(y(x)) dy_{i_1}(x) \dots dy_{i_k}(x)$$

given

$$\omega = \sum_{i_1, \dots, i_k} f_{i_1, \dots, i_k}(y) dy_{i_1} \dots dy_{i_k}.$$

Show that $\phi^*\omega$ is well-defined.

- (39) Compute all the de Rham cohomology groups of S^2 . Then inductively compute all the de Rham cohomology groups of S^n .
- (40) Suppose the manifold M is the disjoint union of manifolds M_1 and M_2 . Then prove that $H_{dR}^k(M) = H_{dR}^k(M_1) \oplus H_{dR}^k(M_2)$.
- (41) Let $\phi : M \rightarrow N$ be a smooth map between manifolds. If $\omega \in \Omega^k(N)$, then prove that $d(\phi^*\omega) = \phi^*(d\omega)$.
- (42) Complete the proof that the short exact sequence of cochain maps $0 \rightarrow \mathcal{A} \xrightarrow{\phi} \mathcal{B} \xrightarrow{\psi} \mathcal{C} \rightarrow 0$ gives rise to a long exact sequence. (This problem should be done carefully and completely — it's worth 30 points.)
- (43) Let M be an oriented manifold of dimension n and $\omega \in \Omega^n(M)$. Let $\{f_\alpha\}$ be a partition of unity subordinate to an open cover $\{U_\alpha\}$. We defined $\int_M \omega = \sum_\alpha \int_{U_\alpha} \phi_\alpha^*(f_\alpha \omega)$, where $\phi_\alpha : U_\alpha \rightarrow \mathbf{R}^n$ are the coordinate functions. Prove that this definition does not depend on the partition of unity or the open cover chosen.
- (44) Using the version of Stokes' Theorem given in class, prove the classical Stokes' Theorem: Let S be a compact, oriented 2-manifold (i.e., a surface) with boundary in \mathbf{R}^3 and let $F = (F_1, F_2, F_3)$ be a smooth vector field defined on a neighborhood of S . Then:

$$\int_S \langle \text{curl } F, n \rangle dA = \int_{\partial S} F_1 dx + F_2 dy + F_3 dz,$$

where n is a unit normal to S , $\langle \cdot, \cdot \rangle$ is the standard inner product, $dA = n_1 dydz + n_2 dzdx + n_3 dxdy$, $\text{curl } F = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$.

The Cauchy Integral Formula. We will now define complex-valued forms on a complex manifold M . Complex-valued differential forms are simply sums $\omega = \omega_1 + i\omega_2$, where ω_1 and ω_2 are real. The wedge product extends in the obvious way, and

$$d\omega \stackrel{\text{def}}{=} d\omega_1 + id\omega_2, \quad \int_M \omega \stackrel{\text{def}}{=} \int_M \omega_1 + i \int_M \omega_2.$$

Note that Stokes' Theorem is valid for complex-valued forms since it is valid separately for the real part and the complex part.

On $\mathbf{C} = \mathbf{R}^2$, let $z = x + iy$ be the complex coordinate and $\bar{z} = x - iy$ be its complex conjugate. Then $dz = dx + idy$ and $d\bar{z} = dx - idy$.

- (45) Consider the function $f : \mathbf{C} \rightarrow \mathbf{C}$, $z \mapsto f(z)$. Show that $\omega = f(z)dz$ is closed ($d\omega = 0$) if and only if $f(z) = f(x, y)$ (f is viewed as $\mathbf{R}^2 \rightarrow \mathbf{R}^2$) satisfies the Cauchy-Riemann

equation

$$\frac{\partial f}{\partial y} = i \frac{\partial f}{\partial x}.$$

Recall that a smooth function f satisfies the Cauchy-Riemann equation if and only if f is holomorphic. Also recall that rational functions in z are holomorphic where defined. For example, $f(z) = \frac{1}{z-a}$ is holomorphic on $\mathbf{C} - \{a\}$.

- (46) Suppose f is a holomorphic function on a domain $\Omega \subset \mathbf{C}$. Prove that if γ_0 and γ_1 are homotopic curves in Ω , then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

Two curves $\gamma_0, \gamma_1 : S^1 \rightarrow \Omega$ are said to be *homotopic* if there exists a smooth $\Gamma : S^1 \times [0, 1] \rightarrow \Omega$ with $\Gamma(\theta, t) = \gamma_t(\theta)$.

- (47) Let C be a circle of radius R around $a \in \mathbf{C}$. Then prove that

$$\int_C \frac{1}{z-a} dz = 2\pi i.$$

- (48) Let $f(z)$ be a holomorphic function on Ω , let $C \subset \Omega$ be a circle of radius R around $a \in \Omega$, and let γ be a smooth closed curve which is homotopic to C inside $\Omega - \{a\}$. Then prove that

$$\int_{\gamma} \frac{f(z)}{z-a} dz = 2\pi i f(a).$$

[Hint: By Exercise (46), the integral only depends on the curve up to homotopy. Therefore, we may assume $\gamma = C$. Shrink the radius R and take the limit.]

- (49) The vector field $(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$ (defined almost everywhere) on \mathbf{R}^2 has curl zero, but it cannot be written as the gradient of any function. Explain what this means in terms of de Rham cohomology.
- (50) Let V be an \mathbf{R} -vector space. Prove that the interior product $i_v : \bigwedge^k V^* \rightarrow \bigwedge^{k-1} V^*$, $v \in V$, where $f_1 \wedge \cdots \wedge f_k \mapsto \sum_l (-1)^{l+1} f_1 \wedge \cdots \wedge f_l(v) \cdots \wedge f_k$, is well-defined.
- (51) Let M be a manifold and X a global vector field on M . Show that $L = d \circ i_X + i_X \circ d : \Omega^k(M) \rightarrow \Omega^k(M)$ is a derivation, i.e., it satisfies $L(\alpha \wedge \beta) = L(\alpha) \wedge \beta + \alpha \wedge L(\beta)$. Here i_X denotes the interior product with X .
- (52) Compute the de Rham cohomology of a compact 2-dimensional manifold (surface) of genus g without boundary. [For this problem, you may feel free to draw pictures of the surface of genus g and its decomposition into pieces. You will probably need to use the Mayer-Vietoris sequence a couple of times, as well as the homotopy properties.] Hint: remove a disk from the surface and see what you are left with.
- (53) For each integer n , exhibit a smooth map $S^1 \rightarrow S^1$ of degree n . (You must prove that your map has degree n .)

- (54) Let M be a manifold. Consider the map $\wedge : H^k(M) \times H^l(M) \rightarrow H^{k+l}(M)$, $([\omega], [\eta]) \mapsto [\omega \wedge \eta]$. Prove that \wedge is well-defined (on the level of cohomology). [Hence $H^*(M) \stackrel{\text{def}}{=} \bigoplus_{i=0}^{\dim M} H^i(M)$ has the structure of an algebra.]
- (55) Let T^2 be the 2-dimensional torus. Compute the map $\wedge : H^1(T^2) \times H^1(T^2) \rightarrow H^2(T^2)$. (Give a basis $\{\omega_1, \dots, \omega_k\}$ for $H^1(T^2)$ and compute $[\omega_i \wedge \omega_j]$.) In particular, prove that \wedge is surjective.
- (56) Prove that every smooth map $f : S^2 \rightarrow T^2$ has degree zero. (Hint: Use the previous exercise and the fact that $H^1(S^2) = 0$.)
- (57) Let M be a compact orientable manifold of dimension $2n$ without boundary, and let ω be a *symplectic form* on M , i.e., a closed 2-form such that $\omega^n = \omega \wedge \dots \wedge \omega$ is nowhere zero. Prove that $H^2(M) \neq 0$.
- (58) Verify that Lie brackets satisfy anticommutativity and the Jacobi identity.
- (59) Prove that if X, X_1, \dots, X_k are vector fields on M and $\omega \in \Omega^k(M)$, then

$$\mathcal{L}_X(\omega(X_1, \dots, X_k)) = (\mathcal{L}_X \omega)(X_1, \dots, X_k) + \sum_i \omega(X_1, \dots, [X, X_i], \dots, X_k).$$

- (60) Prove that if $\omega \in \Omega^k(M)$, and X_1, \dots, X_{k+1} are vector fields on M , then

$$\begin{aligned} d\omega(X_1, \dots, X_{k+1}) &= \sum_i (-1)^{i+1} X_i(\omega(X_1, \dots, \widehat{X}_i, \dots, X_{k+1})) \\ &\quad + \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{k+1}). \end{aligned}$$

Here \widehat{X}_i means omit the term with X_i .

- (61) Consider the vector field $X = x^2 \frac{d}{dx}$ on \mathbf{R} . Compute its integral curves. Explain why X does not admit a global flow $\Phi : \mathbf{R} \times (-\varepsilon, \varepsilon) \rightarrow \mathbf{R}$.
- (62) Find a piecewise smooth curve connecting $(0, 0, 0)$ to any point (x, y, z) , where each smooth piece an integral curve of the 2-plane field distribution $\xi = \ker(xdy - ydx + dz)$. (Such curves are far from unique.)
- (63) Write out the proof of Frobenius' Theorem in the general case.
- (64) Let E be a rank k vector bundle over M which is parallelizable. If s_1, \dots, s_k are global sections which span E_p at every point $p \in M$, then we can write any $s \in \Gamma(E, M)$ as $s = \sum_i f_i s_i$, where $f_i \in C^\infty(M)$. Show $\nabla_X s = \sum_i (X f_i) s_i$ is a connection.
- (65) Let $U \subset M$ be an open subset and let E be a rank k vector bundle over M . Consider two connections ∇' and ∇'' on $E|_U$. If $\lambda_1, \lambda_2 \in C^\infty(U)$ satisfy $\lambda_1 + \lambda_2 = 1$, then prove that $\lambda_1 \nabla' + \lambda_2 \nabla''$ is a connection on $E|_U$.
- (66) Prove that $R(X, Y)s$ is tensorial in X and Y . Here R is the curvature of a connection ∇ and X, Y are vector fields.
- (67) Complete the proof (started in class) that $R = dA + A \wedge A$.
- (68) If ∇ is the Levi-Civita connection on a Riemannian manifold (M, g) and $R = R_\nabla$ is its curvature, then show that R satisfies the following properties:
- (a) $\langle R(X, Y)Z, W \rangle = -\langle R(X, Y)W, Z \rangle$.

- (b) $\langle R(X, Y)Z, W \rangle = \langle R(Z, W)X, Y \rangle$.
- (c) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$.
- (69) Let $\Sigma = \{x^2 + y^2 + z^2 = R^2\} \subset \mathbf{R}^3$, and let \bar{g} be the metric induced from the standard Euclidean metric on \mathbf{R}^3 to Σ . Compute:
- (a) the induced metric \bar{g} and
- (b) the Levi-Civita connection of (Σ, \bar{g}) , locally near $(0, 0, R)$ using the coordinates (x, y) , given by projecting onto the first two coordinates of \mathbf{R}^3 .